

# Algebraic extensions of an Eilenberg-MacLane spectrum in the category of ring spectra

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## 1. Introduction.

The content of the following note lives at the border between algebra and topology. Historically the origin and development of algebraic topology was stimulated by beautiful applications of algebraic methods for solving topological problems. Later it turned out that going in opposite direction can be fruitful for algebra also. Perhaps the first observation of this type can be derived from the celebrated Dold-Kan theorem from the fifties, which can be viewed as a statement that topological observations about Eilenberg-MacLane spaces should have meaning in the category of chain complexes. In proceeding years we observe quick development of the point of view that topological objects and methods should give fruitful observations for algebra. We can give many examples here like various applications in algebra of topologically defined homology theories or algebraic  $K$ -theory, but of course this is not our aim in this paper.

John Rognes in [R] defined extensions of ring spectra which have algebraic origin and flavor. So we can talk about Galois extensions of ring spectra, separable extensions,  $thh$ -étale extensions and just étale ones. When  $R$  is a ring we can associate to it an Eilenberg-MacLane ring spectrum  $HR$  so we can view problems about rings as problems in topology. We would like to spend some time on studying the following question: do we get this way any new extensions of an Eilenberg-MacLane spectrum  $HR$  for a commutative ring  $R$ ? In other words: does every extension of  $HR$  come from an extension of rings of the corresponding type (Galois, separable, étale)? Speaking again in a different way: do we get anything new for the theory of rings via embedding them in the stable homotopy category?

The following note is mostly devoted to the easy part of the problem. We are going to show that in the case of Galois extensions the answer to the question above is negative. So every Galois extension of  $HR$  in the category of spectra comes from Galois extension of rings. Such a strong statement is not true in the case of separable extensions. We show that under some additional assumptions imposed on the extension we get similar statement. Every ring spectrum comes with an associated graded ring of homotopy groups. Hence in both cases we first prove the corresponding statement about graded rings and then about ring spectra. The graded algebraic case is not necessary for topological arguments but can serve as a source of some good intuitions.

In Section 5 of the paper we approach the case of étale type extensions of spectra. Here we can fully answer our main question only in the connective case and the answer is the same as for extensions of Galois type. On the other hand we know that in general the situation for étalé extensions is different than in the connective case. In some sense this was the crucial observation of Mandell, which was the starting point for the consideration of this note. As discussed in [MM, example 3.5], Mandell in private communication showed

that the extension  $HF_p \rightarrow B$  is étale (in certain sense) where  $B = F(K(Z/p, n), HF_p)$  is a mod  $p$  cochain  $HF_p$ -algebra of an Eilenberg-MacLane space  $K(Z/p, n)$  for  $n \geq 2$ .

In the paper we use freely language of [R] and [EKMM]. So while in topological world we work in the category of  $S$ -algebras and  $S$ -modules, where  $S$  of course denotes the sphere spectrum. In algebra all our rings are unital with unital maps.

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## 2. Preliminaries on spectra.

In this short section we try to put all necessary notation needed for the rest of the paper. As was said in the introduction our basic reference is [EKMM]. Let us recall briefly from there what we mean under the word "spectrum". Assume that we have a structure of a real inner product space on  $R^\infty$ . Then a spectrum  $E$  is a way of associating a based space to every finite dimensional vector space  $V \subset R^\infty$  with a structure homeomorphisms

$$\sigma_{V,W} : EV \rightarrow \Omega^{W-V} EW$$

when  $V \subset W$ . Here  $W - V$  is the orthogonal complement of  $V$  in  $W$  and  $\Omega^W X$  is the space of based maps from  $S^W$  to  $X$ , where  $S^W$  is a one point compactification of  $W$ . The map between spectra is just a family of based maps index by  $V \subset R^\infty$  commuting with the structure maps. This way we obtain a category of spectra  $\mathcal{S}$ . The functor from spectra to spaces given by restriction to the  $V$ th space has a left adjoint which is denoted in [EKMM] by  $E_V^\infty$ , or  $E_n^\infty$  in the case when  $V = R^n$ . When  $V = 0$  it is a suspension functor  $E^\infty$ . For  $0 \leq n$  we define the spectrum  $n$ -sphere  $S^n$  as  $E^\infty S^{R^n}$ . For  $0 > n$  we define  $S^n$  as  $E_{-n}^\infty S^0$ . For  $m \geq 0$  there are canonical isomorphisms  $E^m S^{R^n} \simeq S^{m+n}$  and  $E_m^\infty S^n \simeq S^{n-m}$ . We define homotopy groups of a spectrum  $E$  to be

$$\pi_n(E) = h\mathcal{S}(S^n, E)$$

where  $h$  stands here for the homotopy category of spectra, obtained via correct choice of a closed model category structure on  $\mathcal{S}$ .

The letter  $S$  denotes the 0-sphere spectrum, as defined in [EKMM, Section 1], called usually just as sphere spectrum. It comes with a map  $S \wedge S \rightarrow S$  giving it a structure of a ring spectrum. Throughout the paper we are working in the category of  $S$ -algebras, as defined in [EKMM, Section 2 and 3]. It means that we are considering  $S$ -modules, denoted by capital letters  $A, B$ , etc., equipped additionally with the multiplication and unit maps

$$\mu : A \wedge_S A \rightarrow A$$

$$1_A : S \rightarrow A$$

satisfying standard associativity and unity conditions. When  $A$  is an  $S$ -algebra we can define the symmetric monoidal category  $\mathcal{M}_A$  of right  $A$ -modules (left  $A$ -modules  ${}_A\mathcal{M}$ ). As objects in it we have  $S$ -algebras equipped additionally with the right action of  $A$  on them:

$$B \wedge_S A \rightarrow B$$

satisfying standard associativity and unity conditions, which we know from the algebraic category of modules. We say that  $B$  is an  $A$ -algebra if it is a monoid in  $\mathcal{M}_A$  (compare [EKMM, Section 7]). In case  $A$  is not commutative we can also talk about categories of right and left  $A$ -modules but our algebras will be always unital by what we mean that they come equipped with a unit map

$$1_B^A : A \rightarrow B$$

which is compatible with  $1_A$  and  $1_B$ . We consistently remove  $S$  from our notation, hence for example  $1_A$  is the same as  $1_A^S$ ,  $A \wedge A$  denotes  $A \wedge_S A$ , etc., etc.

### 3. Galois extensions.

Let  $A$  and  $B$  are commutative rings and  $G$  a finite group. Following [G] we define:

**Definition 3.1:** We say that the extension of commutative rings  $A \hookrightarrow B$  is  $G$ -Galois if  $G$  is a subgroup of  $\text{Aut}(B/A)$ ,  $B^G = A$  and the map  $h : B \otimes_A B \rightarrow \text{Map}(G, B)$  is a  $B$ -algebra isomorphism, where  $h(x \otimes y)(g) = x \cdot g(y)$ .

In the case of graded rings we assume that the action of  $G$  preserves grading. By a grading we always mean here  $\mathbb{Z}$ -grading. The  $B$ -algebra of functions  $\text{Map}(G, B)$  will be also viewed very often as  $\prod_{g \in G} B$  so we can project from it on the coordinate corresponding to the given  $g \in G$ . Observe that the map  $h$  preserve natural gradings of the source and the target.

**Theorem 3.2:** Let  $A \hookrightarrow B$  be a Galois extension of graded rings. If  $A$  is nontrivial only in grade 0 then the same is true for  $B$ .

Proof. We show first (after [G, Theorem 1.6]) that  $B$  is a finitely generated projective  $A$ -module. The proof given there works as well in the graded case. We present it here because the careful looking at the formulas from topological point of view gives us the desired result for spectra. Let  $\sum x_i \otimes y_i$  be the preimage of  $(1, 0, \dots, 0) \in \prod_{g \in G} B$ , where 1 is at the coordinate corresponding to the unit  $e$  of  $G$ . Define the  $A$ -linear trace  $tr : B \rightarrow A$  by  $tr(y) = \sum_{g \in G} g(y)$ . Let  $\varphi_i : B \rightarrow A$  be defined by  $\varphi_i(z) = tr(zy_i)$ . Then the direct calculation gives us the formula for any  $z \in B$ :

$$(3.2.1) \quad z = \sum_i \varphi_i(z) \cdot x_i$$

This immediately implies that  $B$  is a finitely generated projective  $A$ -module because formula (3.2.1) shows that the pairs  $(x_i, \varphi_i)$  form a dual basis for  $B$  over  $A$ . But for us it is

more important to observe that formula (3.2.1) shows that  $B$  can have non trivial elements only in finitely many gradations (is finitely graded) because we have only finitely many  $x_i$ s and  $A$  is fully in the 0-grade. This observation immediately implies our statement. If  $k$  is the highest (lowest) nontrivial gradation of  $B$  then  $B \otimes_A B$  has highest (lowest) nontrivial gradation in dimension  $2k$ . But grading of  $\prod_{g \in G} B$  is the same as the grading of  $B$ . Hence  $k$  has to be 0.

Now we move towards topology. Let  $A \rightarrow B$  be a map of commutative  $S$ -algebras and  $G$  is a finite group acting continuously from the left on  $B$  via the  $A$ -algebra maps. Let us recall (compare [R, Definition 4.1.3]) the definition of the Galois extensions in the category of  $S$ -algebras.

**Definition 3.3:** With the assumptions as above we say that  $A \rightarrow B$  is a  $G$ -Galois extension of  $S$ -algebras if two canonical maps of  $S$ -modules  $i : A \rightarrow B^{hG}$  and  $h : B \wedge_A B \rightarrow F(G_+, B)$  are weak equivalences.

Perhaps we should also recall here after [R] the definitions of the maps  $i$  and  $h$ . The map  $i : A \rightarrow B^{hG} = F(EG_+, B)^G$  is the right adjoint to the composite  $G$ -equivariant map  $A \wedge EG_+ \rightarrow A \rightarrow B$ , collapsing the contractible free  $G$ -space to a point. The map  $h$  is right adjoint to the composite map  $B \wedge_A B \wedge G_+ \rightarrow B \wedge_A B \rightarrow B$  where the first map come from the action of  $G$  on the middle  $B$  from the left and the second is just the multiplication map. Observe that in our case ( $G$  finite) we can equally well write  $F(G_+, B)$  as  $\prod_{g \in G} B$ . Note also that we can view  $h$  as

$$B \wedge_A B \xrightarrow{id \wedge \prod g} B \wedge_A \prod_{g \in G} B \xrightarrow{\prod \mu \circ (id \wedge pr_g)} \prod_{g \in G} B$$

where we denote by  $g$  the map  $B \rightarrow B$  coming from the action of  $g \in G$  on  $B$  and  $pr_g$  denotes the projection on the  $g$ -factor.

**Theorem 3.4:** Let  $A = HR \rightarrow B$  be a  $G$ -Galois extension of commutative ring spectra. Then  $B$  is equivalent to  $H(\pi_0 B)$  and  $R \rightarrow \pi_0 B$  is a  $G$ -Galois extension of commutative rings.

*Proof.* The proof is a combination of results from [R] and [EKMM]. By [R, Proposition 6.2.1] we know that  $B$  is a dualizable  $A$ -module. Then by [R, Proposition 3.3.3] and [EKMM Chapter III, Theorem 7.9] we know that  $B$  is a retract of a finite cell  $A$ -module. This implies that  $B$  has only finitely many nontrivial homotopy groups each of which is a finitely generated  $R$ -module.

We will prove first that  $B$  is an Eilenberg-MacLane spectrum. We will follow the lines of the algebraic graded case, the argument is only a little more delicate. On the other hand this is the crucial step because the rest of our theorem is then proved in [R, Theorem 4.2.1]. For the readers convenience we will sketch Rognes' argument later.

Let  $k$  be the lowest integer such that  $\pi_k(B) \neq 0$ . Assume that  $k < 0$ . Then by the spectral sequence for the homotopy groups of a smash product, which is described below, we know that in  $\pi_{2k}(B \wedge_A B)$  we have classes coming from  $\pi_k(B) \otimes_R \pi_k(B)$ . If this

latter group is nontrivial we get a contradiction as in the graded algebraic case. But the group in question indeed is nontrivial by a simple algebraic lemma, probably well known to everybody:

**Lemma 3.4.1:** Assume that  $T$  is a commutative ring and  $M$  is a finitely generated module over  $T$ . Then  $M \otimes_T M$  is nontrivial.

Proof. Assume that  $M$  has only one generator. Then  $M$  is isomorphic to  $R/I$  for a certain ideal  $I$ . Let  $J$  be a maximal ideal containing  $I$  then  $R/I$  maps epimorphically onto  $R/J$ . We know that  $R/J \otimes_T R/J$  is nontrivial by maximality of  $J$  (is isomorphic to  $R/J$ ) so by right exactness of the tensor product we know that  $R/I \otimes_T R/J$  is nontrivial. Hence, again by the right-exactness of the tensor product we get that  $R/I \otimes_T R/I$  is nontrivial.

We can proceed further by induction with respect to the number of generators in  $M$ . If  $M$  has  $n$  generators then it fits into an exact sequence of  $T$ -modules

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

in which  $L$  has one and  $N$  has  $n - 1$  generators. By induction  $N \otimes_T N$  is nontrivial and  $M \otimes_T M$  maps epimorphically onto  $M \otimes_T N$  which maps onto  $N \otimes_T N$  by the left-exactness of the tensor product. So the proof of our lemma is finished.

Now we come back to the proof of 3.4. If  $k \geq 0$ , and hence  $B$  is connective, we know by [EKMM IV, Proposition 1.4] that the dual  $A$ -spectrum of  $B$  is coconnective (has nontrivial homotopy groups only in nonpositive dimensions). On the other hand by [R, Proposition 6.4.7]  $B$  is self dual. So homotopy groups of  $B$  have to be concentrated in dimension 0, as we wanted to show.

Now we can finish the proof of 3.4. Since we know now that  $B$  is an Eilenberg-MacLane spectrum we can recall [R, Proposition 4.2.1]. Let us write  $T$  for  $\pi_0(B)$  for shortness. By [EKMM, IV.4.3] we have the homotopy fixed point spectral sequence

$$E_{s,t}^2 = H^{-s}(G, \pi_t HT) \Longrightarrow \pi_{s+t}(HT^{hG})$$

which in our case gives us  $T^G \simeq \pi_0(HT^{hG}) \simeq \pi_0(HR) = R$

Similarly we have useful spectral sequence for the homotopy groups of a smash product, which was used before and will be crucial in the next section. It is of the form

$$E_{s,t}^2 = \text{Tor}_{s,t}^R(T, T) \Longrightarrow \pi_{s+t}(HT \wedge_{HR} HT)$$

It gives  $T \otimes_R T \simeq \pi_0(HT \wedge_{HR} HT) \simeq \pi_0(\prod_{g \in G} HT) = \prod_{g \in G} T$ . This implies that  $R \rightarrow T$  is  $G$ -Galois in the algebraic sense.

**Remark 3.5:** The proof that  $B_*$  is finitely generated over  $R$  is not direct. We would like to present below a sketch of a direct argument which mimics the proof of the algebraic equivalent statement used in the proof of 3.2.

Our extension is  $G$ -Galois so the map  $h$  defined before is a weak equivalence. This means that the unit map  $1_B : S \rightarrow B$  can be factored as

3.5.1

$$S \xrightarrow{\varphi} B \wedge_A B \xrightarrow{h} \prod_{g \in G} B \xrightarrow{pr_e} B$$

where  $\varphi$  is the preimage of  $(1, 0, \dots, 0) \in \pi_0(\prod_{g \in G} B)$  and, as was defined before,  $pr_e$  is the projection map on the coordinate corresponding to the trivial element  $e \in G$ . Equivalently, by the choice of  $\varphi$ , we could say that  $1_B$  can be factored as

3.5.2

$$S \xrightarrow{\varphi} B \wedge_A B \xrightarrow{h} \prod_{g \in G} B \xrightarrow{\oplus 1_B} B$$

Let  $f : S^n \rightarrow B$  be a map representing an element in  $\pi_n(B)$ . Then

$$S \wedge S^n \xrightarrow{1_B \wedge f} B \wedge_A B \xrightarrow{\mu} B$$

represents the same element in  $\pi_n(B)$  as  $f$ . But instead of  $1_B$  we can use the composition of maps from 3.5.1 or 3.5.2. The composition  $B \xrightarrow{\Delta} \prod_{g \in G} B \xrightarrow{\oplus g} B$  will be denoted by  $\phi$  in the future. Observe that the following two maps:

3.5.3

$$B \wedge_A B \wedge_A B \xrightarrow{h \wedge id} (\prod_{g \in G} B) \wedge_A B \xrightarrow{(\oplus id) \wedge id} B \wedge_A B \xrightarrow{\mu} B$$

and

3.5.4

$$B \wedge_A B \wedge_A B \xrightarrow{id \wedge \mu} B \wedge_A B \xrightarrow{id \wedge \phi} B \wedge_A B \xrightarrow{\mu} B$$

are homotopic after precomposing with  $\varphi \wedge f$ . This follows immediately from the definition of  $\varphi$ . The map 3.5.3 precomposed with  $\varphi \wedge f$  is homotopic to  $f$ . On the other hand the map  $\phi$ , as being  $G$ -invariant, factors through the spectrum  $B^{hG}$  which is equivalent to  $A$ . Hence the homotopy properties of 3.5.4 precomposed with  $\varphi \wedge f$  depend only on the homotopy class of  $\varphi$  and homotopy groups of  $A$ . This implies immediately that  $B$  can have only finitely many nontrivial homotopy groups.

#### 4. Separable extensions.

For separable extensions of ring spectra we would like to prove the same statement as was proved for Galois extensions in the previous section. From the ideological point of view this is the expected statement because as in algebra one expects that any commutative separable extension embeds into a  $G$ -Galois one, for a certain  $G$ . We are able to get the

expected result under additional hypothesis on the extension. We expect that this result is well known to experts but we could not find any place with a proof written down. But before going into stable homotopy category let us state and prove the graded algebraic counterpart of this statement. Later we will generalize the proof to the case of spectra. Let  $A$  and  $B$  be two  $\mathbf{Z}$ -graded unital rings.

**Definition 4.1:** We say that  $A \rightarrow B$  is separable if the  $A$ -algebra multiplication map  $\mu : B \otimes_A B^{op} \rightarrow B$ , considered as a map in the category of  $B$ -bimodules, admits a section  $\sigma : B \rightarrow B \otimes_A B^{op}$ .

**Theorem 4.2:** Assume that  $A$  is concentrated in gradation 0 only. Let  $A \rightarrow B$  be a separable extension of graded rings as defined above and  $B$  has no zero divisors in the subring  $B_0$ . Then  $B$  is concentrated in gradation 0.

Proof. The crucial but obvious observation in the case  $A = A_0$  is that if  $x_1 \otimes x_2 = x_3 \otimes x_4 \neq 0$  in  $B \otimes_A B$  and all  $x'_i$ s are of homogeneous degree then  $\deg(x_1) = \deg(x_3)$  and  $\deg(x_2) = \deg(x_4)$ . This is the case because  $B \otimes_A B$  has double grading and multiplication by elements of  $A$  preserves it. Separability means that there exists an element

$$\sum_{i=1}^k b_i \otimes c_i \in (B \otimes B)_0 \subset B \otimes_A B$$

satisfying for any  $b \in B$

$$b \left( \sum_{i=1}^k b_i \otimes c_i \right) = \left( \sum_{i=1}^k b_i \otimes c_i \right) b$$

The element described above is equal to the image of 1 under the map  $\sigma : B \rightarrow B \otimes_A B$ . We can assume that the elements  $b_i$  and  $c_i$  are homogeneous and  $\deg(c_i) = -\deg(b_i)$ . Let  $\{b_{i_j}\}_{j \in J}$  be the set of these  $b_i$ s which have the highest grade. Then  $b(\sum_{j \in J} b_{i_j} \otimes c_{i_j}) = 0$  for any  $b$  of grade bigger than 0 by degree reasons. The same one can say about any  $b$  of negative degree considering  $(\sum_{j \in J} b_{i_j} \otimes c_{i_j})b = 0$ . Hence either  $b$  has to be zero or by our assumption on  $B_0$ ,  $\sum_{j \in J} b_{i_j} \cdot c_{i_j} = 0$  and we can send 1 to

$$\sum_{i=1}^k b_i \otimes c_i - \sum_{j \in J} b_{i_j} \otimes c_{i_j}$$

But this latter element has the lower highest degree among  $b_i$ s so step by step we can lower this highest degree in our sum to 0. Obviously then all  $c_i$ s should have also degree 0. This means, we can assume that  $\sigma(1) \in B_0 \otimes B_0$ . But then in order to have satisfied

$$b \left( \sum_{i=1}^k b_i \otimes c_i \right) = \left( \sum_{i=1}^k b_i \otimes c_i \right) b$$

an element  $b$  cannot have degree different from 0. Thus  $B$  should have only 0 grade and  $A_0 \rightarrow B_0$  should be a separable extension of ungraded rings.

**Remark 4.3:** It is easy to observe that if our rings are only  $\mathbf{N}$ -graded (connective) then theorem 4.2 is true without any assumption on  $B_0$ .

Now we come to the definition of separable extension of ring spectra, as it is given in [R, Definition 9.1.1].

**Definition 4.4:** We say that  $A \rightarrow B$  is separable if the  $A$ -algebra multiplication map  $\mu : B \wedge_A B^{op} \rightarrow B$ , considered as a map in the stable homotopy category of  $B$ -bimodules relative to  $A$ , admits a section  $\sigma : B \rightarrow B \wedge_A B^{op}$ .

Observe that in our case if  $B$  is an extension of  $HR$  then being a module over an Eilenberg-MacLane spectrum it is equivalent in the stable homotopy category to the wedge of Eilenberg-MacLane spectra  $H(B_i, i)$ . Of course here  $B_i = \pi_i(B)$ , every  $B_i$  carries a structure of an  $R$ -module and hence  $H(B_i, i)$  carries a structure of  $HR$ -module as well. Let  $v : \bigvee H(B_i, i) \rightarrow B$  gives us an equivalence guaranteed above.

**Theorem 4.5:** Let  $HR \rightarrow B$  be a separable extension as defined above. Assume, similarly as previously, that  $\pi_*(B)$  has no 0-divisors in  $\pi_0(B)$ . Assume moreover that the map  $v$  described above is an  $HR$ -module map. Then  $B$  is equivalent to  $H\pi_0 B$  and  $R \rightarrow \pi_0 B$  is a separable extension of rings.

Proof. We would like to follow the lines of the proof of 4.2 taking as an extension of  $R$  the ring  $B_* = \pi_*(B)$ . The map  $\sigma$  gives us the splitting of  $\pi_*(B)$  from  $\pi_*(B \wedge_A B^{op})$ . But we are not able to use 4.2 directly because the statement  $\pi_*(B \wedge_A B^{op}) = \pi_*(B) \otimes_{\pi_*(A)} \pi_*(B)$  is false in general. Instead, as it was mentioned in the previous section, we have only a spectral sequence converging to  $\pi_*(B \wedge_A B^{op})$  ([EKMM, chapter IV] with the second table given by the formula

$$E_{p,q}^2 = Tor_p^{\pi_*(A)}(B_*, B_*)_q$$

In order to apply similar argument as previously we are forced to study the bimodule (kind of) structure of this second table over  $B_*$ . The main point is that  $\sigma$  is a bimodule map so for every element  $b \in \pi_*(B)$  we have as previously

$$\sigma_*(b) = b\sigma_*(1) = \sigma_*(1)b$$

Hence the multiplication by  $b$  on  $\sigma_*(1) \in \pi_*(B \wedge_A B)$  should have the same effect when we use right and left module structures. Our ground ring spectrum is  $HR$  hence every group  $B_i$  is a module over  $R$  and graded  $R$ -resolution of  $B_*$  is just a graded sum of ordinary  $R$ -resolutions of  $B_i$ s. This leads to the splitting formula for  $Tor$ -groups:

$$Tor_p^R(B_*, B_*) = \bigoplus_{i,j} Tor_p^R(B_i, B_j)$$

and

$$E_{p,q}^2 = \bigoplus_{i+j=q} Tor_p^R(B_i, B_j)$$

Moreover, because of our hypothesis on the map  $v$ ,  $B$  is homotopically a wedge of Eilenberg-MacLane spectra itself. Hence we can write a resolution of  $B_*$  coming from the wedge of resolutions of  $H(B_i, i)$ s. This leads to the resolution of  $B_*$  which is a sum of resolutions of



$B_i$ s. Then the spectral sequence above can be viewed as a sum of spectral sequences coming from  $\pi_*(S^i \wedge HB_i \wedge B)$ . This implies also that the limit group splits into a corresponding sum.

Let  $b$  be represented by a map  $S^n \rightarrow B$ , which will be called also  $b$ . Then the multiplication by  $b$  on  $B_*$  from the left can be described by a map

$$S^n \wedge B \rightarrow B \wedge B \rightarrow B \wedge_{HR} B \rightarrow B$$

where the first map is  $b \wedge id$ , the next comes from the fact that  $S$ -module and  $HR$ -module structures on  $B$  are compatible, the last map is  $\mu$  and  $\wedge$  without subscript denotes smashing over the sphere spectrum. Of course we have similar description for the right multiplication. We can smash the sequence above with  $B$  over  $HR$  and get a natural map  $S^n \wedge B \wedge_{HR} B \rightarrow B \wedge_{HR} B$ . Similarly we have a natural map  $B \wedge_{HR} B \wedge S^n \rightarrow B \wedge_{HR} B$  and these two maps give us two maps of spectral sequences (for the functoriality of the spectral sequence construction see [EKMM, Section IV.5])

$${}_l E_{p,q}^2 = Tor_p^R(\pi_*(S^n \wedge B), \pi_*(B))_q \rightarrow Tor_p^R(\pi_*(B), \pi_*(B))_q = E_{p,q}^2$$

and

$${}_r E_{p,q}^2 = Tor_p^R(\pi_*(B), \pi_*(B \wedge S^n))_q \rightarrow Tor_p^R(\pi_*(B), \pi_*(B))_q = E_{p,q}^2$$

where the letters  $l$  and  $r$  refer to the left and right multiplication by  $b$ . Observe that  $\pi_*(S^n \wedge B \wedge B) = \pi_{*-n}(B \wedge B)$  as one is an  $n$ -fold suspension of another. Spectral sequences  ${}_l E_{p,q}^2$  and  ${}_r E_{p,q}^2$  are the same as  $E_{p,q}^2$  with a shift of total grading by  $n$ . The described above map  ${}_l E_{p,q}^2 \rightarrow E_{p,q}^2$  is induced by multiplication with  $b$  and on the level of the algebraic second table of spectral sequences it is induced by the map  $\cdot b : B_{*-n} \rightarrow B_*$  on the first variable in the groups  $Tor$ . One has the same description for the map  ${}_r E_{p,q}^2 \rightarrow E_{p,q}^2$  but the multiplication goes along the second variable in the  $Tor$ -groups. So starting from  $Tor_p(B_i, B_j)$  we go by multiplication by  $b$  one time to  $Tor_p(B_{i+n}, B_j)$  and the second to  $Tor_p(B_i, B_{j+n})$ . When we know all of this we can apply the same procedure as in 4.2 to the image of  $1 \in \pi_0(B)$  in  $\pi_*(B \wedge_{HR} B)$ .

Multiplication from the left by an element  $b \in B_n$  takes  $B_i$  to  $B_{i+n}$  and this is a map of right  $R$ -modules hence extends to the map of resolutions. It means that multiplication with  $b$  from the left induces a map of spectral sequences

$${}_i E_{p,q}^2 \rightarrow {}^{i+n} E_{p,q}^2$$

On the other hand multiplication from the right by  $b$  is a map of coefficients which is a map of left  $R$  modules. It means it induces self maps of  ${}_i E_{p,q}^2$ 's. Now we can easily argue that if  $b$  has grade different from 0 then multiplication by it on  $\sigma_*(1) \in \pi_*(B \wedge_{HR} B)$  should be trivial. As was stated before  $\sigma_*(1)$  can be decomposed into a finite sum of nontrivial elements  $\sum_{j=1}^k x_{i_j}$ , where each piece  $x_{i_j}$  comes from  ${}_i E_{p,q}^2$  for different  $i_j$ 's. Let  $k$  be the highest among  $i_j$ 's. Then  $b \cdot \sigma_*(1)$  has a summand coming from  ${}^{k+n} E_{p,q}^2$ . On the other hand  $\sigma_*(1) \cdot b$  does not have such a summand. It means that  $b \cdot x_k$  should be 0 for any  $b$  of degree higher than 0. Observe that by definition  $x_k$  is of homotopical degree 0 so by our hypothesis on  $B_*$  if  $b$  is non zero then  $x_k$  must be 0. This way we can lower the

maximal index  $k$  in the decomposition of  $\sigma_*(1)$ . But this argument works always when  $B_*$  has elements of degree higher than 0 so we know that  $B_*$  should be concentrated in non-positive degrees. But of course for  $b$  of negative degree we argue similarly, starting from  $x_m$  with the lowest possible index  $m$ . This implies that  $B$  is an Eilenberg-MacLane spectrum and the extension  $HR \rightarrow B$  comes from a separable extension of  $R \rightarrow R'$  where obviously  $B \simeq HR'$ .

**Example 4.6**(after Birgit Richter): Let  $B = F_{HR}(\Sigma HR \vee \Sigma^{-1} HR, \Sigma HR \vee \Sigma^{-1} HR)$ , where  $F(.,.)$  denotes the mapping spectrum in appropriate category. This is a Brauer-trivial Azumaya algebra over  $HR$ , so in particular it is separable over  $HR$ . Observe that  $B_*(= \pi_*(B))$  is isomorphic to  $R$  for  $* = 2, -2$  and  $B_0 = R \oplus R$ . One checks directly that  $B_*$  is isomorphic as an  $R$ -algebra to the algebra of  $2 \times 2$ -matrices over  $R$  with appropriate grading. So  $B_*$  is separable over  $R$ . Hence it is difficult to imagine how one could make our assumption on  $B_0$  in 4.2 or 4.5 weaker.

## 5. Étale extensions.

Let us start this section from the definition of the topological Hochschild homology(see [R, section 9.2] or [EKMM, Chapter IX]):

**Definition 5.1:** Let  $B$  be an algebra over a commutative  $S$ -algebra  $A$ . Then we define

$$THH^A(B) = Tor^{B \wedge_A B^{op}}(B, B)$$

**Definition 5.2:** We say that  $A \rightarrow B$  is formally symmetrically étale if the canonical map  $\zeta : B \rightarrow THH^A(B)$  is a weak equivalence.

For an extension  $A \rightarrow B$  of ordinary rings we should use the same definition of topological Hochschild homology as above for an extension  $HA \rightarrow HB$ . This is because  $THH$ -theory does not have algebraic definition. But of course this leads to problems with the graded algebraic case because we do not know what is  $HB$  for a graded ring  $B$ . Define  $THH_i^A(B) = \pi_i(THH^A(B))$ , which has precise meaning in the ungraded case. Then  $THH_1^A(B) = HH_1(B)$  where  $HH$  denotes the ordinary Hochschild homology. Hochschild homology theory has perfect meaning also in the graded case and so we can say something about the graded rings using Hochschild homology groups. Assume that  $B$  is a graded ring which is an algebra over a commutative ring  $A$ . We treat  $A$  as a graded object concentrated in gradation 0. Moreover assume that  $B$  is commutative and trivial in negative gradations. Then

**Lemma 5.3:** If  $B$  is nontrivial in positive gradations and  $B_0 = A$  then  $HH_1(B)$  is nontrivial. Hence we can think about  $B$  as being not formally symmetrically étale over  $A$ .

Proof: The first Hochschild homology group of  $B$  is the same as the group of Kähler differentials of  $B$  over  $A$ . If  $B_k$  is the lowest positive nontrivial gradation of  $B$  then  $(\Omega_{B/A}^1) \supset B_k \neq 0$ . This follows directly from the definition of Kähler differentials. If

$b \in B_k$  is of the form  $b_1 \cdot b_2$  then either  $b_1 \in A$  or  $b_2 \in A$  by our choice of  $k$ . Hence there are no relations except linearity over  $A$  between generators  $db$  of  $(\Omega_{B/A}^1)$  for the elements  $b \in B_k$ .

The lemma above shows the way in which we can approach the similar problem for ring spectra. The key ingredient is hidden in the spectrum  $\Omega_{B/A}$  of differential forms of  $B$  over  $A$  defined as a cofibrant replacement of the homotopy fiber of the multiplication map  $\mu : B \wedge_A B^{op} \rightarrow B$ . Assume that  $B$  is a connective  $HA$ -algebra where  $HA$  is an Eilenberg-MacLane spectrum of a commutative ring  $A$ . Assume that  $\pi_0(B) = A$ . Then

**Theorem 5.4:** If  $B$  has higher nontrivial homotopy groups then  $B$  is not a formally symmetrically étale extension of  $HA$ .

Proof. Let  $k$  be the smallest natural number bigger than 0 for which  $\pi_k(B) \neq 0$ . Then from the spectral sequence for  $\pi_*(B \wedge_{HA} B)$  we know that  $\pi_k(B \wedge_{HA} B) = \pi_k(B) \oplus \pi_k(B)$ . Observe that there is a map

$$i : B \rightarrow B \wedge_{HA} B$$

which composed with multiplication

$$\mu : B \wedge_{HA} B \rightarrow B$$

is trivial. It is the topological counterpart of the algebraic map  $X \rightarrow X \otimes X$  which takes  $x \in X$  to  $x \otimes 1 - 1 \otimes x$ . The map  $i$  is defined as the difference of maps  $id_B \wedge 1_B$  and  $1_B \wedge id_B$ . Of course  $\mu \circ i = 0$  and hence  $i$  factors as  $j \circ \beta$  through the homotopy fiber  $j : \Omega_{B/A} \rightarrow B \wedge_{HA} B$  of  $\mu$ . Let  $\alpha : S^k \rightarrow B$  represents the nontrivial element in  $\pi_k(B)$ . Then  $i \circ \alpha$  is nontrivial on homotopy groups by the formula for  $\pi_k(B \wedge_{HA} B)$  and hence  $\beta \circ \alpha$  is also nontrivial. It means that  $\Omega_{B/A}$  is not contractible and has nontrivial  $k$ -th homotopy group. We know that  $B$ ,  $B \wedge_{HA} B^{op}$  and  $\Omega_{B/A}$  are connective and it follows that the latter spectrum is  $(k-1)$ -connected from the definition of  $\Omega_{B/A}$ . Moreover we know that  $\pi_0(B) = A = \pi_0(B \wedge_{HA} B^{op})$  and  $\pi_i(B \wedge_{HA} B^{op}) = 0$  for  $i = 1, \dots, k-1$ . This last calculation follows directly from the spectral sequence for calculating homotopy groups of  $B \wedge_{HA} B^{op}$ .

Again, by the definition of  $\Omega_{B/A}$  we have a cofiber sequence

$$B \wedge_{B \wedge_{HA} B^{op}} \Omega_{B/A} \rightarrow B \xrightarrow{\zeta} THH^{HA}(B)$$

Our proof will be finished if we show that  $B \wedge_{B \wedge_{HA} B^{op}} \Omega_{B/A}$  is not weakly equivalent to a point. But again we can use the spectral sequence for calculating homotopy groups of this spectrum with the second table given by the formula

$$E_{s,t}^2 = Tor_{s,t}^{\pi_*(B \wedge_{HA} B^{op})}(B, \Omega_{B/A})$$

Taking into account the connectivity of  $\Omega_{B/A}$  and assumptions on  $B$  we immediately read that  $\pi_k(B \wedge_{B \wedge_{HA} B^{op}} \Omega_{B/A})$  equals to the  $k$ th grade of  $\pi_*(B) \otimes_{\pi_*(B \wedge_{HA} B^{op})} \pi_*(\Omega_{B/A})$ . But

this latter group is easily calculated by dimension reasons as  $A \otimes_A \pi_k(\Omega_{B/A}) = \pi_k(\Omega_{B/A}) \neq 0$ .

**Remark 5.5:** Observe that in both Lemma 5.3 and Theorem 5.4 above the assumption on the 0-grade is irrelevant. In 5.3 by the described above arguments one gets the result for differential forms of  $B$  over  $B_0$ . But then either  $0 \neq (\Omega_{B_0/A}^1) \subset (\Omega_{B/A}^1)$  or  $(\Omega_{B_0/A}^1) = 0$  and then by the same argument as previously we get  $(\Omega_{B/A}^1) \supset B_k \neq 0$ . The argument for 5.4 is left to the interested reader.

**Remark 5.6:** In the commutative case Rognes defined the notion of formally étale extension  $A \rightarrow B$  using the notion of topological André-Quillen homology. We do not recall it here because the property of being formally symmetrically étale is equivalent to formally étale for connective algebras. So far we are not able to analyze the non-connective cases, where we know that these two notions are different by Mandell's example.

**Cojecture 5.7:** Theorem 5.4 is true for  $HA$ -algebras which are bounded below.

As an evidence we we show below that we can extend the proof of the lemma 5.3 to cover algebras which are bounded below. We have the following lemma:

**Lemma 5.8:** Assume that  $B$  is a graded commutative  $A$ -algebra,  $B_0 = A$  and  $B_i = 0$  for  $i < k$  where  $k$  is some negative number. Then  $HH_1(B)$  is nontrivial.

Proof: As previously we will use the fact that  $HH_1(B)$  is equal to the  $B$ -module of Kähler differentials and the latter module is the same as  $I/I^2$  where  $I$  is a kernel of the multiplication map  $B \otimes_A B \rightarrow B$ . Assume that  $x \in B_k$  and  $dx = x \otimes 1 - 1 \otimes x$  is in  $I^2$ . Then by the argument from the beginning of the proof of 4.2 and the fact that  $k$  is the minimal grade with nontrivial  $B_k$  we get immediately that  $x$  can be expressed as an  $A$ -combination of elements of the form  $x_1 \cdot x_2$  where both  $x_1$  and  $x_2$  have negative grading. So if for every  $x \in B_k$  the differential  $dx$  is trivial in  $HH_1(B)$  then all elements of  $B_k$  are sums of multiples of elements of higher but negative degree. We can extend this reasoning easily to other negative degrees of  $B$  by induction and get that if the differential  $dx = 0$  for  $x \in B_s$ ,  $s$  negative, then  $x$  can be expressed as a sum of finite multiples of elements of negative but higher than  $s$  degree. So either we have nontrivial elements of negative degree in  $HH_1(B)$  or negative part of  $B$  is generated over  $A$  by  $B_t$  where  $B_t$  is the highest lower than 0 nontrivial grade of  $B$ . But if this is the case then the only relations among  $dx$  for  $x \in B_t$  are relations of  $A$  linearity. Hence we have nontrivial elements in degree  $t$  of  $HH_1(B)$ .

**Remark 5.9:** Assume that  $B$  is a bounded below  $HA$ -algebra with  $\pi_0(B) = A$  and for any  $i$ ,  $\pi_i(B)$  is a projective  $A$ -module. Then one can easily apply the way of reasoning from the proof of 5.8 to showing that theorem 5.4 can be extended to cover such a case.

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